

Segal–Sugawara vectors for the Lie algebra of type G_2

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Abstract

Explicit formulas for Segal–Sugawara vectors associated with the simple Lie algebra \mathfrak{g} of type G_2 are found by using computer-assisted calculations. This leads to a direct proof of the Feigin–Frenkel theorem describing the center of the corresponding affine vertex algebra at the critical level. As an application, we give an explicit solution of Vinberg’s quantization problem by providing formulas for generators of maximal commutative subalgebras of $U(\mathfrak{g})$. We also calculate the eigenvalues of the Hamiltonians on the Bethe vectors in the Gaudin model associated with \mathfrak{g} .

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1 Introduction

For a simple Lie algebra \mathfrak{g} over \mathbb{C} equipped with a standard symmetric invariant bilinear form, consider the corresponding (non-twisted) affine Kac–Moody algebra $\widehat{\mathfrak{g}}$

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K. \quad (1.1)$$

The universal vacuum module $V(\mathfrak{g})$ over $\widehat{\mathfrak{g}}$ is the quotient of $U(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$. A *Segal–Sugawara vector* is any element $S \in V(\mathfrak{g})$ with the property

$$\mathfrak{g}[t]S \in (K + h^\vee)V(\mathfrak{g}),$$

where h^\vee is the dual Coxeter number for \mathfrak{g} . In particular, the canonical quadratic Segal–Sugawara vector is given by

$$S = \sum_{a=1}^d X_a[-1]^2, \quad (1.2)$$

where X_1, \dots, X_d is an orthonormal basis of \mathfrak{g} and we write $X[r] = Xt^r$ for $X \in \mathfrak{g}$.

By an equivalent approach, Segal–Sugawara vectors are elements of the subspace $\mathfrak{z}(\widehat{\mathfrak{g}})$ of invariants of the vacuum module at the critical level

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g})_{\text{cri}} \mid \mathfrak{g}[t]v = 0\}, \quad (1.3)$$

where $V(\mathfrak{g})_{\text{cri}}$ is the quotient of $V(\mathfrak{g})$ by the submodule $(K + h^\vee)V(\mathfrak{g})$. The vacuum module possesses a vertex algebra structure, and by the definition (1.3), $\mathfrak{z}(\widehat{\mathfrak{g}})$ coincides with the *center* of the vertex algebra $V(\mathfrak{g})_{\text{cri}}$. This induces a structure of commutative associative algebra on the center which coincides with the one obtained via identification of $\mathfrak{z}(\widehat{\mathfrak{g}})$ with a subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$.

The structure of $\mathfrak{z}(\widehat{\mathfrak{g}})$ was described by a theorem of Feigin and Frenkel [8], which states that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials in infinitely many variables; see [11] for a detailed exposition of these results. The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is referred to as the *Feigin–Frenkel center*. Explicit formulas for generators of this algebra were found in [5] for type A and in [17] for types B , C and D ; see also [4] and [20] for simpler arguments in type A and extensions to Lie superalgebras.

Our goal in this paper is to give explicit formulas for generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ in the case where \mathfrak{g} is the exceptional Lie algebra of type G_2 . In particular, we obtain a direct proof of the Feigin–Frenkel theorem in this case. Furthermore, using the connections with the Gaudin model as discovered in [9], we get formulas for higher Gaudin Hamiltonians associated with \mathfrak{g} and calculate their eigenvalues on the Bethe vectors; see also [10]. In the classical types such formulas were given in a recent work [19].

As another application, following [10] and [25] we give explicit formulas for algebraically independent generators of maximal commutative subalgebras of $U(\mathfrak{g})$. These subalgebras

\mathcal{A}_μ are parameterized by regular elements $\mu \in \mathfrak{g}^*$, and their classical limits $\overline{\mathcal{A}}_\mu$ are Poisson commutative subalgebras of $S(\mathfrak{g})$ known as the Mishchenko–Fomenko or shift of argument subalgebras. The formulas for generators of \mathcal{A}_μ thus provide an explicit solution of Vinberg’s quantization problem [27].

Our calculations of the explicit expressions for the Segal–Sugawara vectors and their Harish-Chandra images were computer-assisted. We gratefully acknowledge the use of the *Symbolic Manipulation System FORM* originally developed by Vermaseren [26].

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2 Lie algebra of type G_2 and its matrix presentation

We start by recalling well-known matrix presentations of an arbitrary simple Lie algebra \mathfrak{g} ; see, e.g., [6] and [16].

2.1 Matrix presentations of simple Lie algebras

Equip \mathfrak{g} with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Choose a basis X^1, \dots, X^d of \mathfrak{g} and let X_1, \dots, X_d be its dual with respect to the form. Let π be a faithful representation of \mathfrak{g} afforded by a finite-dimensional vector space V ,

$$\pi : \mathfrak{g} \rightarrow \text{End } V. \quad (2.1)$$

Introduce the elements

$$G = \sum_{i=1}^d \pi(X^i) \otimes X_i \in \text{End } V \otimes U(\mathfrak{g}) \quad (2.2)$$

and

$$\Omega = \sum_{i=1}^d \pi(X^i) \otimes \pi(X_i) \in \text{End } V \otimes \text{End } V. \quad (2.3)$$

Note that G and Ω are independent of the choice of the basis X^i . In particular,

$$\Omega = \sum_{i=1}^d \pi(X_i) \otimes \pi(X^i). \quad (2.4)$$

Consider the tensor product algebra $\text{End } V \otimes \text{End } V \otimes U(\mathfrak{g})$ and identify Ω with the element $\Omega \otimes 1$. Also, introduce its elements

$$G_1 = \sum_{i=1}^d \pi(X^i) \otimes 1 \otimes X_i \quad \text{and} \quad G_2 = \sum_{i=1}^d 1 \otimes \pi(X^i) \otimes X_i. \quad (2.5)$$

Write the commutation relations for \mathfrak{g} ,

$$[X_i, X_j] = \sum_{k=1}^d c_{ij}^k X_k \quad (2.6)$$

with structure coefficients c_{ij}^k . We will regard the universal enveloping algebra $U(\mathfrak{g})$ as the associative algebra with generators X_i subject to the defining relations (2.6), where the left hand side is understood as the commutator $X_i X_j - X_j X_i$.

Proposition 2.1. *The defining relations of $U(\mathfrak{g})$ are equivalent to the matrix relation*

$$G_1 G_2 - G_2 G_1 = -\Omega G_2 + G_2 \Omega. \quad (2.7)$$

Proof. The left hand side of (2.7) reads

$$\sum_{i,j=1}^d \pi(X^i) \otimes \pi(X^j) \otimes (X_i X_j - X_j X_i).$$

For the right hand side we have

$$-\sum_{i,k=1}^d \pi(X^i) \otimes \pi([X_i, X^k]) \otimes X_k. \quad (2.8)$$

By the invariance of the form, we find

$$\langle [X_i, X^k], X_j \rangle = -\langle X^k, [X_i, X_j] \rangle = -c_{ij}^k.$$

Hence (2.8) equals

$$\sum_{i,j,k=1}^d c_{ij}^k \pi(X^i) \otimes \pi(X^j) \otimes X_k.$$

Since the representation π is faithful, we conclude that (2.7) is equivalent to the defining relations (2.6) of $U(\mathfrak{g})$. \square

The defining relations (2.7) can be written in an equivalent form

$$G_1 G_2 - G_2 G_1 = \Omega G_1 - G_1 \Omega, \quad (2.9)$$

which is easily verified with the use of (2.4). The element G can be regarded as an $n \times n$ matrix ($n = \dim V$) with entries in $U(\mathfrak{g})$. We point out another well-known property of this matrix which goes back to [14].

Corollary 2.2. *All elements $\text{tr } G^k$ with $k \geq 1$ belong to the center of $U(\mathfrak{g})$.*

Proof. Relation (2.7) implies

$$G_1 G_2^k - G_2^k G_1 = -\Omega G_2^k + G_2^k \Omega.$$

By taking trace over the second copy of $\text{End } V$ and using its cyclic property, we get $[G_1, \text{tr } G_2^k] = 0$ as required. \square

The Casimir elements $\text{tr } G^k$ are widely used in representation theory, especially for the Lie algebras \mathfrak{g} of classical types. In those cases one usually takes V to be the first fundamental (or vector) representation.

Note also that the relation (2.7) can be regarded as the ‘classical part’ of the *RTT* presentation of the Yangian $Y(\mathfrak{g})$ associated with \mathfrak{g} ; see [6]. More precisely, $Y(\mathfrak{g})$ contains $U(\mathfrak{g})$ as a subalgebra, and (2.7) is recovered as a reduction of the *RTT* relation to the generators of this subalgebra.

2.2 Lie algebra of type G_2

The simple Lie algebra \mathfrak{g} of type G_2 admits a few different presentations; see, e.g., [12], [28]. It is well-known that it can be embedded into the orthogonal Lie algebras \mathfrak{o}_7 and \mathfrak{o}_8 ; these embeddings were employed in [21] to construct the classical \mathcal{W} -algebra for \mathfrak{g} . We will follow [12, Lect. 22] to realize \mathfrak{g} as the direct sum of vector spaces

$$\mathfrak{g} = \mathbb{C}^3 \oplus \mathfrak{sl}_3 \oplus (\mathbb{C}^3)^*.$$

The Lie bracket on \mathfrak{g} is determined by the conditions that \mathfrak{sl}_3 is a subalgebra of \mathfrak{g} , the vector spaces \mathbb{C}^3 and $(\mathbb{C}^3)^*$ are, respectively, the vector representation of \mathfrak{sl}_3 and its dual, together with additional brackets

$$\mathbb{C}^3 \times \mathbb{C}^3 \rightarrow (\mathbb{C}^3)^*, \quad (\mathbb{C}^3)^* \times (\mathbb{C}^3)^* \rightarrow \mathbb{C}^3 \quad \text{and} \quad \mathbb{C}^3 \times (\mathbb{C}^3)^* \rightarrow \mathfrak{sl}_3.$$

To produce a matrix presentation of \mathfrak{g} as provided by Proposition 2.1, consider the 7-dimensional representation $\pi : \mathfrak{g} \rightarrow \text{End } V$ with $V \cong \mathbb{C}^7$ where the action is described explicitly as follows. Write an arbitrary element of \mathfrak{g} as a triple (v, A, φ) , where $A \in \mathfrak{sl}_3$ is a traceless 3×3 matrix,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{C}^3 \quad \text{and} \quad \varphi = [\varphi_1, \varphi_2, \varphi_3] \in (\mathbb{C}^3)^*.$$

Then, under the representation π we have

$$\pi : (v, A, \varphi) \mapsto \begin{bmatrix} A & v & \frac{1}{\sqrt{2}}B(\varphi^t) \\ \varphi & 0 & -v^t \\ \frac{1}{\sqrt{2}}B(v) & -\varphi^t & -A^t \end{bmatrix},$$

where t denotes the antidiagonal matrix transposition,

$$v^t = [v_3, v_2, v_1], \quad \varphi^t = \begin{bmatrix} \varphi_3 \\ \varphi_2 \\ \varphi_1 \end{bmatrix}, \quad (A^t)_{ij} = A_{4-j, 4-i}$$

and

$$B(v) = \begin{bmatrix} v_2 & -v_1 & 0 \\ -v_3 & 0 & v_1 \\ 0 & v_3 & -v_2 \end{bmatrix}.$$

The representation π is faithful, so we may use it to identify the Lie algebra \mathfrak{g} with its image under π where elements of \mathfrak{g} can be regarded as 7×7 matrices. Then the bilinear form on \mathfrak{g} defined by

$$\langle X, Y \rangle = \frac{1}{6} \operatorname{tr} XY \quad (2.10)$$

is symmetric and invariant. Note that this form is proportional to the standard normalized Killing form

$$\frac{1}{2h^\vee} \operatorname{tr} (\operatorname{ad} X \operatorname{ad} Y),$$

where $h^\vee = 4$ is the dual Coxeter number for \mathfrak{g} . We have

$$\langle X, Y \rangle = \frac{1}{24} \operatorname{tr} (\operatorname{ad} X \operatorname{ad} Y).$$

The additional scalar factor $1/3$ is meant to simplify our formulas for Segal–Sugawara vectors by avoiding fractions.

Let $e_{ij} \in \operatorname{End} \mathbb{C}^7$ denote the standard matrix units. For all $1 \leq i, j \leq 7$ set $f_{ij} = e_{ij} - e_{j'i'}$, where $i' = 8 - i$. The following elements form a basis of \mathfrak{g} :

$$f_{11} - f_{22}, \quad f_{22} - f_{33}, \quad f_{ij} \quad \text{with} \quad 1 \leq i, j \leq 3 \quad \text{and} \quad i \neq j,$$

together with

$$f_{14} - \frac{1}{\sqrt{2}} f_{3'2}, \quad f_{24} - \frac{1}{\sqrt{2}} f_{1'3}, \quad f_{34} - \frac{1}{\sqrt{2}} f_{2'1},$$

and

$$f_{41} - \frac{1}{\sqrt{2}} f_{23'}, \quad f_{42} - \frac{1}{\sqrt{2}} f_{31'}, \quad f_{43} - \frac{1}{\sqrt{2}} f_{12'}.$$

In the general setting of Sec. 2.1, these elements are understood as the basis X^1, \dots, X^{14} . The elements X_1, \dots, X_{14} of the dual basis with respect to the form (2.10) are then given by the following expressions, where we use the corresponding capital letters to think of the X_i as abstract generators of $U(\mathfrak{g})$ rather than matrices:

$$2F_{11} - F_{22} - F_{33}, \quad F_{11} + F_{22} - 2F_{33}, \quad 3F_{ji} \quad \text{with} \quad 1 \leq i, j \leq 3 \quad \text{and} \quad i \neq j,$$

together with

$$2F_{41} - \sqrt{2}F_{23'}, \quad 2F_{42} - \sqrt{2}F_{31'}, \quad 2F_{43} - \sqrt{2}F_{12'},$$

and

$$2F_{14} - \sqrt{2}F_{3'2}, \quad 2F_{24} - \sqrt{2}F_{1'3}, \quad 2F_{34} - \sqrt{2}F_{2'1}.$$

Using (2.2), we can now define the entries G_{ij} of the matrix G from the expansion

$$G = \sum_{i,j=1}^7 e_{ji} \otimes G_{ij} \in \text{End } \mathbb{C}^7 \otimes \text{U}(\mathfrak{g}). \quad (2.11)$$

In particular,

$$G_{11} = 2F_{11} - F_{22} - F_{33}, \quad G_{22} = 2F_{22} - F_{11} - F_{33}, \quad G_{33} = 2F_{33} - F_{11} - F_{22},$$

so that $G_{11} + G_{22} + G_{33} = 0$. Also, for all $1 \leq i, j \leq 3$ with $i \neq j$ we have $G_{ij} = 3F_{ij}$. Furthermore,

$$G_{14} = 2F_{14} - \sqrt{2}F_{3'2}, \quad G_{24} = 2F_{24} - \sqrt{2}F_{1'3}, \quad G_{34} = 2F_{34} - \sqrt{2}F_{2'1},$$

and

$$G_{41} = 2F_{41} - \sqrt{2}F_{23'}, \quad G_{42} = 2F_{42} - \sqrt{2}F_{31'}, \quad G_{43} = 2F_{43} - \sqrt{2}F_{12'}.$$

The remaining entries of the matrix G are determined by the symmetry properties $G^t = -G$ which give $G_{ij} + G_{j'i'} = 0$ together with

$$G_{14} = -\sqrt{2}G_{3'2}, \quad G_{24} = -\sqrt{2}G_{1'3}, \quad G_{34} = -\sqrt{2}G_{2'1},$$

and

$$G_{41} = -\sqrt{2}G_{23'}, \quad G_{42} = -\sqrt{2}G_{31'}, \quad G_{43} = -\sqrt{2}G_{12'}.$$

Note that the above formulas define an explicit embedding of \mathfrak{g} into the orthogonal Lie algebra \mathfrak{o}_7 spanned by the elements $F_{ij} = E_{ij} - E_{j'i'}$, where the E_{ij} denote the standard basis elements of \mathfrak{gl}_7 . An expression for the element Ω defined in (2.3) can be given by

$$\begin{aligned} \Omega = & 3 \sum_{i,j=1}^3 f_{ij} \otimes f_{ji} - \sum_{i,j=1}^3 f_{ii} \otimes f_{jj} + 2 \sum_{i=1}^3 (f_{4i} \otimes f_{i4} + f_{i4} \otimes f_{4i}) \\ & + \odot_{1,2,3} (f_{12'} \otimes f_{2'1} + f_{2'1} \otimes f_{12'}) \\ & - \sqrt{2} \odot_{1,2,3} (f_{14} \otimes f_{23'} + f_{23'} \otimes f_{14} + f_{41} \otimes f_{3'2} + f_{3'2} \otimes f_{41}), \end{aligned} \quad (2.12)$$

where the symbol $\odot_{1,2,3}$ indicates the summation over cyclic permutations of the indices 1, 2, 3 keeping all other symbols, including primes, at their positions; that is,

$$\odot_{1,2,3} X_{1,2',3} = X_{1,2',3} + X_{3,1',2} + X_{2,3',1}.$$

Proposition 2.1 provides a matrix form (2.7) of the defining relations of $\text{U}(\mathfrak{g})$ with the elements G and Ω defined in (2.11) and (2.12).

2.3 A formula for Ω as an element of the centralizer algebra

By its definition (2.3), the element Ω can be viewed as an operator

$$\Omega : V \otimes V \rightarrow V \otimes V.$$

It is easily seen that this operator commutes with the action of the Lie algebra \mathfrak{g} on $V \otimes V$ given by

$$X \mapsto \pi(X) \otimes 1 + 1 \otimes \pi(X).$$

This implies that Ω must be a linear combination of the projections of $V \otimes V$ onto its irreducible components.

As a representation of the Lie algebra \mathfrak{sl}_7 , the tensor product $\mathbb{C}^7 \otimes \mathbb{C}^7$ of two copies of $V = \mathbb{C}^7$ splits into the direct sum of two irreducible components

$$\mathbb{C}^7 \otimes \mathbb{C}^7 = \Lambda^2(\mathbb{C}^7) \oplus S^2(\mathbb{C}^7),$$

afforded by the exterior and symmetric square of V . The canonical projections onto the irreducible components are given by the respective operators $(1 - P)/2$ and $(1 + P)/2$, where P is the permutation operator

$$P = \sum_{i,j=1}^7 e_{ij} \otimes e_{ji}.$$

The restriction of the representation $\Lambda^2(\mathbb{C}^7)$ to the subalgebra \mathfrak{o}_7 remains irreducible, whereas the restriction of $S^2(\mathbb{C}^7)$ splits into two irreducible components; each of them remains irreducible under the further restriction to the subalgebra $\mathfrak{g} \subset \mathfrak{o}_7$ of type G_2 :

$$S^2(\mathbb{C}^7) \cong V_0 \oplus V_{2\omega_1}.$$

The respective projections are given by the operators $Q/7$ and $(1 + P)/2 - Q/7$, where Q is the operator

$$Q = \sum_{i,j=1}^7 e_{ij} \otimes e_{i'j'}.$$

It is obtained by applying the antidiagonal transposition

$$t : \text{End } \mathbb{C}^7 \rightarrow \text{End } \mathbb{C}^7, \quad (e_{ij})^t = e_{j'i'},$$

to the first or the second component of the permutation operator, $Q = P^{t_1} = P^{t_2}$. Here V_0 is the trivial one-dimensional representation of \mathfrak{g} , and $V_{2\omega_1}$ is the 27-dimensional representation corresponding to the double of the first fundamental weight ω_1 .

When restricted to \mathfrak{g} , the exterior square $\Lambda^2(\mathbb{C}^7)$ splits into two irreducible components

$$\Lambda^2(\mathbb{C}^7) \cong V_{\omega_1} \oplus V_{\omega_2}$$

of the respective dimensions 7 and 14, associated with the fundamental weights. The respective projections are given by the operators $T/6$ and $(1 - P)/2 - T/6$, where the operator T is defined by means of a 3-form on \mathbb{C}^7 as follows. The 3-form β is defined in the canonical basis e_1, \dots, e_7 of \mathbb{C}^7 by

$$\beta = \sum_{i=1}^3 e_i \wedge e_4 \wedge e_{i'} + \sqrt{2} e_1 \wedge e_2 \wedge e_3 + \sqrt{2} e_{3'} \wedge e_{2'} \wedge e_{1'}.$$

One easily verifies that β is invariant under the action of \mathfrak{g} , which provides a \mathfrak{g} -module embegging $\mathbb{C}^7 \hookrightarrow \Lambda^2(\mathbb{C}^7)$ defined as contraction with β . Explicitly, the operator T can be written as

$$T = \sum_{i,j,k,l=1}^7 \sum_{a=1}^7 \beta_{ika} \beta_{jla} e_{ij} \otimes e_{kl},$$

where the coefficients of the form β are defined by

$$\beta = \sum_{i,j,k=1}^7 \beta_{ijk} e_i \otimes e_j \otimes e_k$$

via the standard embedding $\Lambda^3(\mathbb{C}^7) \hookrightarrow (\mathbb{C}^7)^{\otimes 3}$ such that

$$e_{i_1} \wedge e_{i_2} \wedge e_{i_3} = \sum_{\sigma \in \mathfrak{S}_3} \text{sgn } \sigma \cdot e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes e_{i_{\sigma(3)}}.$$

Thus, we have an expansion

$$\Omega = a \Pi_0 + b \Pi_{\omega_1} + c \Pi_{\omega_2} + d \Pi_{2\omega_1}$$

into a linear combination of the projections to the respective irreducible components of the decomposition

$$\mathbb{C}^7 \otimes \mathbb{C}^7 \cong V_0 \oplus V_{\omega_1} \oplus V_{\omega_2} \oplus V_{2\omega_1}.$$

The coefficients in the expansion are found by applying Ω to particular vectors to give the formula

$$\Omega = 1 + P - 2Q - T. \quad (2.13)$$

Since $\Pi_0, \Pi_{\omega_1}, \Pi_{\omega_2}$ and $\Pi_{2\omega_1}$ are pairwise orthogonal projections, we derive the following relations for the operators P, Q and T . They commute pairwise and

$$P^2 = 1, \quad Q^2 = 7Q, \quad T^2 = 6T, \quad PQ = Q, \quad PT = -T, \quad QT = 0. \quad (2.14)$$

By (2.13) the matrix form (2.7) of the defining relations implies a uniform expression for the commutators of the generators of \mathfrak{g} ,

$$[G_{ij}, G_{kl}] = \delta_{kj} G_{il} - \delta_{il} G_{kj} - 2\delta_{ki'} G_{j'l} + 2\delta_{j'l} G_{ki'} + \sum_{a,b=1}^7 \left(\beta_{iab} \beta_{jlb} G_{ka} - \beta_{ikb} \beta_{jab} G_{al} \right). \quad (2.15)$$

Remark 2.3. It is known by Ogievetsky [22] that a rational R -matrix associated with \mathfrak{g} can be given by the formula

$$R(u) = 1 - \frac{P}{u} + \frac{2Q}{u-6} + \frac{T}{u-4}.$$

Its expansion into a power series in u^{-1} takes the form

$$R(u) = 1 - (\Omega - 1) u^{-1} + \dots$$

so that the classical limit of the RTT relations defining the Yangian $Y(\mathfrak{g})$ reproduces the defining relations (2.7) for $U(\mathfrak{g})$; see [6].

2.4 Isomorphism with the Chevalley presentation

The simple Lie algebra \mathfrak{g} of type G_2 is associated with the Cartan matrix $A = [a_{ij}]$,

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

Its Chevalley presentation is defined by generators e_i, h_i, f_i with $i = 1, 2$, subject to the defining relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, & [h_i, h_j] &= 0, \\ [h_i, e_j] &= a_{ij} e_j, & [h_i, f_j] &= -a_{ij} f_j, \end{aligned}$$

together with (the Serre relations)

$$(\text{ad } e_1)^2 e_2 = 0, \quad (\text{ad } e_2)^4 e_1 = 0, \quad (\text{ad } f_1)^2 f_2 = 0, \quad (\text{ad } f_2)^4 f_1 = 0.$$

We let α and β denote the simple roots. The set of positive roots is

$$\alpha, \quad \beta, \quad \alpha + \beta, \quad \alpha + 2\beta, \quad \alpha + 3\beta, \quad 2\alpha + 3\beta.$$

For each positive root γ we let e_γ and f_γ denote the root vectors associated with γ and $-\gamma$, respectively. In particular, $e_1 = e_\alpha$ and $e_2 = e_\beta$. We have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad (2.16)$$

where the subalgebras \mathfrak{n}_+ and \mathfrak{n}_- are spanned by the positive and negative root vectors, respectively, while \mathfrak{h} is the Cartan subalgebra with the basis elements h_1 and h_2 .

By the symmetry properties of the matrix G described in Sec. 2.2, the Lie algebra \mathfrak{g} is spanned by the elements G_{ij} with $1 \leq i, j \leq 4$. Moreover, due to the relations $G_{11} + G_{22} + G_{33} = 0$ and $G_{44} = 0$ this leaves a 14-dimensional Lie algebra. Explicit commutators between these elements can be obtained by using the embedding $\mathfrak{g} \hookrightarrow \mathfrak{o}_7$ defined in Sec. 2.2 or by applying the commutation relations (2.15). Assuming that the indices i, j, k, l run over the set $\{1, 2, 3\}$, the commutation relations of \mathfrak{g} are then completely described as follows:

$$\begin{aligned} [G_{ij}, G_{kl}] &= 3\delta_{kj}G_{il} - 3\delta_{il}G_{kj}, \\ [G_{ij}, G_{k4}] &= 3\delta_{kj}G_{i4} - \delta_{ij}G_{k4}, \\ [G_{ij}, G_{4l}] &= -3\delta_{il}G_{4j} + \delta_{ij}G_{4l}, \\ [G_{i4}, G_{4l}] &= 2G_{il}, \end{aligned}$$

together with the relations

$$[G_{i4}, G_{j4}] = 2\sqrt{2}G_{4k}, \quad [G_{4i}, G_{4j}] = -2\sqrt{2}G_{k4};$$

the latter hold for the triples (i, j, k) of the form $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$.

Proposition 2.4. *The mapping*

$$e_1 \mapsto \frac{1}{3}G_{12}, \quad e_2 \mapsto \frac{1}{\sqrt{2}}G_{24}, \quad f_1 \mapsto \frac{1}{3}G_{21}, \quad f_2 \mapsto \frac{1}{\sqrt{2}}G_{42}$$

defines an isomorphism between the Chevalley and the matrix presentations of \mathfrak{g} . Moreover, under this isomorphism,

$$h_1 \mapsto \frac{1}{3}(G_{11} - G_{22}) \quad \text{and} \quad h_2 \mapsto G_{22}.$$

Proof. This follows easily from the commutation relations for the G_{ij} . □

By definition, the simple root subspaces \mathfrak{g}_γ of \mathfrak{g} are $\mathfrak{g}_\alpha = \langle G_{12} \rangle$ and $\mathfrak{g}_\beta = \langle G_{24} \rangle$, so that by Proposition 2.4 the remaining root subspaces associated with positives roots are spanned by the following elements:

$$\mathfrak{g}_{\alpha+\beta} = \langle G_{14} \rangle, \quad \mathfrak{g}_{\alpha+2\beta} = \langle G_{43} \rangle, \quad \mathfrak{g}_{\alpha+3\beta} = \langle G_{23} \rangle, \quad \mathfrak{g}_{2\alpha+3\beta} = \langle G_{13} \rangle.$$

Similarly, $\mathfrak{g}_{-\alpha} = \langle G_{21} \rangle$ and $\mathfrak{g}_{-\beta} = \langle G_{42} \rangle$ together with

$$\mathfrak{g}_{-\alpha-\beta} = \langle G_{41} \rangle, \quad \mathfrak{g}_{-\alpha-2\beta} = \langle G_{34} \rangle, \quad \mathfrak{g}_{-\alpha-3\beta} = \langle G_{32} \rangle, \quad \mathfrak{g}_{-2\alpha-3\beta} = \langle G_{31} \rangle.$$

Therefore, in the triangular decomposition (2.16) we have

$$\mathfrak{n}_+ = \langle G_{12}, G_{24}, G_{14}, G_{43}, G_{23}, G_{13} \rangle \quad \text{and} \quad \mathfrak{n}_- = \langle G_{21}, G_{42}, G_{41}, G_{34}, G_{32}, G_{31} \rangle. \quad (2.17)$$

3 Invariants of the vacuum module

Suppose that \mathfrak{g} is an arbitrary simple Lie algebra equipped with an invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ which is determined uniquely, up to a nonzero factor. We will need matrix presentations of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ analogous to those given in Proposition 2.1 for \mathfrak{g} . The Lie algebra $\widehat{\mathfrak{g}}$ is defined as the central extension (1.1), where $\mathfrak{g}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in \mathfrak{g} . The commutation relations have the form

$$[X[r], Y[s]] = [X, Y][r + s] + r \delta_{r, -s} \langle X, Y \rangle K, \quad X, Y \in \mathfrak{g}, \quad (3.1)$$

and the element K is central in $\widehat{\mathfrak{g}}$.

3.1 Matrix presentations of affine Kac–Moody algebras

We will keep the data associated with the Lie algebra \mathfrak{g} , chosen in the beginning of Sec. 2.1. In particular, we will use a faithful representation (2.1) of \mathfrak{g} and the element Ω defined in (2.3). In addition to the matrix G defined in (2.2), for any $r \in \mathbb{Z}$ introduce the matrix $G[r]$ by

$$G[r] = \sum_{i=1}^d \pi(X^i) \otimes X_i[r] \in \text{End } V \otimes U(\widehat{\mathfrak{g}}). \quad (3.2)$$

By analogy with (2.5) introduce the elements of the algebra $\text{End } V \otimes \text{End } V \otimes U(\widehat{\mathfrak{g}})$ by

$$G_1[r] = \sum_{i=1}^d \pi(X^i) \otimes 1 \otimes X_i[r] \quad \text{and} \quad G_2[r] = \sum_{i=1}^d 1 \otimes \pi(X^i) \otimes X_i[r].$$

Proposition 3.1. *The defining relations of $U(\widehat{\mathfrak{g}})$ are equivalent to the matrix relation*

$$G_1[r]G_2[s] - G_2[s]G_1[r] = -\Omega G_2[r + s] + G_2[r + s]\Omega + r \delta_{r, -s} \Omega K. \quad (3.3)$$

Proof. The proof is completed as for Proposition 2.1, where we use (3.1) and the additional observation that the sum

$$\sum_{i,j=1}^d \langle X_i, X_j \rangle \pi(X^i) \otimes \pi(X^j).$$

coincides with Ω . □

3.2 Segal–Sugawara vectors

We will now use the notation \mathfrak{g} for the simple Lie algebra of type G_2 and follow [11] to recall some general facts on the affine vertex algebra associated with \mathfrak{g} . We equip \mathfrak{g} with the bilinear form (2.10).

The universal enveloping algebra at the critical level $U(\widehat{\mathfrak{g}})_{\text{cri}}$ is the quotient of $U(\widehat{\mathfrak{g}})$ by the ideal generated by $K + 12$. Define the vacuum module $V(\mathfrak{g})_{\text{cri}}$ at the critical level over $\widehat{\mathfrak{g}}$ as the quotient of $U(\widehat{\mathfrak{g}})_{\text{cri}}$ by the left ideal generated by $\mathfrak{g}[t]$. The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined as the subspace of invariants of the vacuum module; see (1.3). Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector. As we pointed out in the Introduction, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative associative algebra which can be regarded as a subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$. The algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$ is equipped with the derivation $D = -d/dt$ whose action on the generators is given by

$$D : X[r] \mapsto -r X[r-1], \quad X \in \mathfrak{g}, \quad r < 0. \quad (3.4)$$

The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $U(t^{-1}\mathfrak{g}[t^{-1}])$ is D -invariant. By [8] (see also [11]) we have the following.

Feigin–Frenkel theorem. *There exist Segal–Sugawara vectors S_2 and S_6 such that all elements $D^k S_2$ and $D^k S_6$ with $k \geq 0$ are algebraically independent, and every element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a polynomial in the $D^k S_2$ and $D^k S_6$.* \square

We call a pair S_2, S_6 satisfying the conditions of the theorem a *complete set* of Segal–Sugawara vectors for \mathfrak{g} . It is known that the elements S_2 and S_6 must have the degrees 2 and 6 with respect to the canonical filtration of $U(t^{-1}\mathfrak{g}[t^{-1}])$ (thus justifying the notation). These are the degrees of algebraically independent generators of the algebra of \mathfrak{g} -invariants of the symmetric algebra $S(\mathfrak{g})$. The vector S_2 of degree 2 must be proportional to the canonical Segal–Sugawara vector (1.2), whereas no formula has been known for a vector of degree 6. Our first main result is an explicit formula for such a vector (see Theorem A below). We keep the notation G_{ij} for the elements of the Lie algebra \mathfrak{g} as defined in Sec. 2. In particular, in accordance with (3.2), for any $r \in \mathbb{Z}$ we have

$$G[r] = \sum_{i,j=1}^7 e_{ji} \otimes G_{ij}[r] \in \text{End } \mathbb{C}^7 \otimes U(\widehat{\mathfrak{g}}). \quad (3.5)$$

Set

$$S_2 = \text{tr } G[-1]^2, \quad (3.6)$$

$$\begin{aligned} S_6 = & \text{tr } G[-1]^6 + 5 \text{tr } G[-2] G[-1]^4 + 14 \text{tr } G[-3] G[-1]^3 + 456 \text{tr } G[-3]^2 \\ & - 639 \text{tr } G[-2]^3 + 31 \text{tr } G[-2]^2 G[-1]^2 - 312 \text{tr } G[-3] G[-2] G[-1]. \end{aligned} \quad (3.7)$$

Theorem A. *The elements S_2 and S_6 form a complete set of Segal–Sugawara vectors for the Lie algebra \mathfrak{g} .*

Proof. Since \mathfrak{g} is a simple Lie algebra, the Lie algebra $\mathfrak{g}[t]$ is generated by all elements $G_{ij}[0]$ together with one nonzero element $G_{kl}[1]$. Therefore, in order to verify the property $S_2, S_6 \in \mathfrak{z}(\widehat{\mathfrak{g}})$, it will be sufficient to demonstrate that in the vacuum module, $G_{ij}[0] S_a = 0$ for all i, j and $G_{11}[1] S_a = 0$ for $a = 2, 6$. The first part of these relations will be implied by the following lemma.

Lemma 3.2. *For any negative integers r_1, \dots, r_p , we have the relations*

$$G_1[0] \operatorname{tr} G_2[r_1] \dots G_2[r_p] = 0$$

in the algebra

$$\operatorname{End} \mathbb{C}^7 \otimes \operatorname{End} \mathbb{C}^7 \otimes \operatorname{U}(\widehat{\mathfrak{g}}), \quad (3.8)$$

where the elements in $\operatorname{U}(\widehat{\mathfrak{g}})$ are considered modulo the left ideal $\operatorname{U}(\widehat{\mathfrak{g}}) \mathfrak{g}[t]$ and the trace is taken with respect to the second copy of $\operatorname{End} \mathbb{C}^7$.

Proof. By (3.3),

$$G_1[0] G_2[s] - G_2[s] G_1[0] = -\Omega G_2[s] + G_2[s] \Omega.$$

The argument is completed in the same way as for Corollary 2.2. \square

Lemma 3.3. *For all $k \geq 1$ we have the relations*

$$G_1[1] \operatorname{tr} G_2[-1]^k = \sum_{i=1}^k \operatorname{tr} \left(G_2[-1]^{i-1} \Omega G_2[-1]^{k-i} \Omega - \Omega G_2[-1]^{i-1} \Omega G_2[-1]^{k-i} \right)$$

in the algebra (3.8) modulo the left ideal of $\operatorname{U}(\widehat{\mathfrak{g}})$ generated by $K + 12$ and $\mathfrak{g}[t]$, where the trace is taken with respect to the second copy of $\operatorname{End} \mathbb{C}^7$.

Proof. Applying (2.9) and (3.3) we find

$$G_1[1] \operatorname{tr} G_2[-1]^k = \sum_{i=1}^k \operatorname{tr} G_2[-1]^{i-1} \left(\Omega G_1[0] - G_1[0] \Omega + K \Omega \right) G_2[-1]^{k-i}.$$

Furthermore,

$$G_1[0] G_2[-1]^{k-i} = -\Omega G_2[-1]^{k-i} + G_2[-1]^{k-i} \Omega,$$

and by applying the partial transposition t_2 to both sides we also get

$$G_1[0] (G_2[-1]^{k-i})^t = -\Omega (G_2[-1]^{k-i})^t + (G_2[-1]^{k-i})^t \Omega$$

since $\Omega^{t_2} = -\Omega$ as implied by (2.12). Hence, using the general property

$$\operatorname{tr} AB = \operatorname{tr} A^t B^t, \quad (3.9)$$

by applying t_2 to the factors, we get

$$\begin{aligned}\mathrm{tr} G_2[-1]^{i-1} G_1[0] \Omega G_2[-1]^{k-i} &= -\mathrm{tr} (G_2[-1]^{i-1})^t G_1[0] (G_2[-1]^{k-i})^t \Omega \\ &= \mathrm{tr} (G_2[-1]^{i-1})^t \Omega (G_2[-1]^{k-i})^t \Omega - \mathrm{tr} (G_2[-1]^{i-1})^t (G_2[-1]^{k-i})^t \Omega^2.\end{aligned}$$

By applying (3.9) again we can write this as

$$\mathrm{tr} \Omega G_2[-1]^{i-1} \Omega G_2[-1]^{k-i} - \mathrm{tr} G_2[-1]^{i-1} (\Omega^2)^{t_2} G_2[-1]^{k-i}.$$

Bringing the calculations together, we obtain

$$\begin{aligned}G_1[1] \mathrm{tr} G_2[-1]^k &= \sum_{i=1}^k \mathrm{tr} \left(G_2[-1]^{i-1} \Omega G_2[-1]^{k-i} \Omega - \Omega G_2[-1]^{i-1} \Omega G_2[-1]^{k-i} \right) \\ &\quad + \sum_{i=1}^k \mathrm{tr} G_2[-1]^{i-1} \left((\Omega^2)^{t_2} - \Omega^2 + K \Omega \right) G_2[-1]^{k-i}.\end{aligned}$$

Since $P^{t_2} = Q$ and $Q^{t_2} = P$, the relation $\Omega^{t_2} = -\Omega$ implies $T^{t_2} = 2 - P - Q - T$, and so we derive from (2.14) that $(\Omega^2)^{t_2} = \Omega^2 + 12\Omega$, thus completing the proof. \square

It is immediate from Lemma 3.3 that S_2 is a Segal–Sugawara vector. In principle, the lemma can be useful for checking this property of S_6 as well; however, this leads to rather cumbersome expressions which we were unable to handle without computer’s assistance. We used the Symbolic Manipulation System FORM; see [26]. Namely, we verified the relation $G_{11}[1]S_6 = 0$ in the vacuum module by employing a program within the FORM which works as follows. We fix a total ordering \prec on the basis elements $G_{ij}[r]$ of \mathfrak{g} with $1 \leq i, j \leq 4$ (excluding $(i, j) = (3, 3)$ and $(4, 4)$) with the property that $G_{ij}[r] \prec G_{kl}[s]$ for $r < s$. The input of the program is the element $G_{11}[1]S_6$ written explicitly as a linear combination of monomials in the generators $G_{ij}[r]$. The output is a linear combination of the ordered monomials which is calculated with the use of the commutation relations (3.3) written in terms of the $G_{ij}[r]$. The program confirms that the last factor of each monomial in this linear combination is of the form $G_{ij}[0]$ or $G_{ij}[1]$.

To complete the proof of the theorem, note that the symbols \overline{S}_2 and \overline{S}_6 of the elements S_2 and S_6 in the associated graded algebra $S(t^{-1}\mathfrak{g}[t^{-1}]) \cong \mathrm{gr} U(t^{-1}\mathfrak{g}[t^{-1}])$ are given by

$$\overline{S}_2 = \mathrm{tr} \overline{G}[-1]^2 \quad \text{and} \quad \overline{S}_6 = \mathrm{tr} \overline{G}[-1]^6,$$

where we use the bar notation to indicate objects associated with the symmetric algebras. It is easily seen (e.g., by taking the Chevalley images) that the elements $\mathrm{tr} \overline{G}^2$ and $\mathrm{tr} \overline{G}^6$ are algebraically independent generators of the algebra of \mathfrak{g} -invariants $S(\mathfrak{g})^{\mathfrak{g}}$. Thus, the elements \overline{S}_2 and \overline{S}_6 are the respective images of $\mathrm{tr} \overline{G}^2$ and $\mathrm{tr} \overline{G}^6$ under the embedding $S(\mathfrak{g}) \hookrightarrow S(t^{-1}\mathfrak{g}[t^{-1}])$ taking $X \in \mathfrak{g}$ to $X[-1]$. Therefore, by a general argument of [11, Sect. 3.5.1], the elements S_2 and S_6 form a complete set of Segal–Sugawara vectors. \square

Proposition 3.4. *The elements*

$$\begin{aligned} S_3 &= \operatorname{tr} G[-1]^3, \\ S_4 &= \operatorname{tr} G[-1]^4 + \operatorname{tr} G[-2] G[-1]^2, \\ S_5 &= \operatorname{tr} G[-1]^5 + \operatorname{tr} G[-2] G[-1]^3, \end{aligned}$$

are Segal–Sugawara vectors for \mathfrak{g} .

Proof. For the element S_3 this follows from Lemma 3.3 with $k = 3$ by the application of (3.9). The claim for S_4 can also be verified with the use of Lemma 3.3 and some additional arguments which we will omit. In fact, for all three elements the claim is also verified by the same computer program as for S_6 ; see the proof of Theorem A. \square

By Theorem A, each of the vectors S_3 , S_4 and S_5 is a polynomial in the $D^k S_2$ and $D^k S_6$. In particular, a direct argument shows that $S_3 = -3 D S_2$. Indeed, first we note the easily verified relation

$$(G[-1]^2)^t = G[-1]^2 + 12 G[-2],$$

so that by applying (3.9) we find

$$S_3 = \operatorname{tr} (G[-1]^2)^t G[-1]^t = -S_3 - 12 \operatorname{tr} G[-2] G[-1].$$

This gives $S_3 = -6 \operatorname{tr} G[-2] G[-1]$ which equals $-3 D \operatorname{tr} G[-1]^2$ since

$$\operatorname{tr} G[-2] G[-1] = \operatorname{tr} G[-1] G[-2].$$

We will be able to write down the remaining polynomials after calculating the Harish–Chandra images of the Segal–Sugawara vectors S_a ; see Corollary 4.2 below.

4 Affine Harish–Chandra isomorphism

It was proved in [8] that for any simple Lie algebra \mathfrak{g} the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is isomorphic to the classical \mathcal{W} -algebra $\mathcal{W}({}^L \mathfrak{g})$ associated with the Langlands dual Lie algebra ${}^L \mathfrak{g}$ (corresponding to the transposed of the Cartan matrix of \mathfrak{g}). We will follow [11, Sec. 8.1] to recall these results before applying them to a particular case of type G_2 .

4.1 Feigin–Frenkel center and classical \mathcal{W} -algebra

Suppose that $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a triangular decomposition for a simple Lie algebra \mathfrak{g} . Regard \mathfrak{h} as a subalgebra of $\widehat{\mathfrak{g}}$ via the embedding taking $H \in \mathfrak{h}$ to $H[0]$. The adjoint

action of \mathfrak{h} on $t^{-1}\mathfrak{g}[t^{-1}]$ extends to the universal enveloping algebra, and we have the homomorphism for the \mathfrak{h} -centralizer

$$\mathfrak{f} : U(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{h}} \rightarrow U(t^{-1}\mathfrak{h}[t^{-1}]) \quad (4.1)$$

which is the projection to the first summand in the direct sum decomposition

$$U(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{h}} = U(t^{-1}\mathfrak{h}[t^{-1}]) \oplus J,$$

where J is the intersection of the centralizer with the left ideal of $U(t^{-1}\mathfrak{g}[t^{-1}])$ generated by $t^{-1}\mathfrak{n}_-[t^{-1}]$. The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of the centralizer, and the restriction of the homomorphism (4.1) to $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields an affine version of the Harish-Chandra isomorphism:

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{W}({}^L\mathfrak{g}), \quad (4.2)$$

where the classical \mathcal{W} -algebra $\mathcal{W}({}^L\mathfrak{g})$ is defined as a subalgebra of $U(t^{-1}\mathfrak{h}[t^{-1}])$ which consists of the elements annihilated by the screening operators; see [11, Theorem 8.1.5]. The algebra $\mathcal{W}({}^L\mathfrak{g})$ is known to possess algebraically independent families of generators. Their explicit form in the classical types goes back to [1] and [15] in type A and to [7] in types B , C and D . For type G_2 such generators were calculated in [21]; see also [3], where they appear in a different context. The images of explicit generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under the isomorphism (4.2) are found in [4] and [5] in type A and in [18] for types B , C and D ; see also [24] for a direct calculation for the Pfaffian-type vector. In the next section we calculate the images of the Segal–Sugawara vectors in type G_2 constructed in Sec. 3.2.

4.2 Harish-Chandra images

Now we assume \mathfrak{g} is of type G_2 and use its Chevalley presentation; see Sec. 2.4. Note that the Cartan matrix A is not symmetric. Although the Langlands dual ${}^L\mathfrak{g}$ is a Lie algebra isomorphic to \mathfrak{g} , this leads to a difference between the forms of the screening operators associated with A and its transpose. In accordance with [11, Sec. 8.1.2], the classical \mathcal{W} -algebra $\mathcal{W}({}^L\mathfrak{g})$ is a subalgebra of $U(t^{-1}\mathfrak{h}[t^{-1}])$ which is defined as the intersection of the kernels of the screening operators V_1 and V_2 . The operators are given by

$$V_i = \sum_{r=0}^{\infty} V_{ir} \sum_{j=1}^2 a_{ji} \frac{\partial}{\partial h_j[-r-1]}, \quad (4.3)$$

where $h_j[s] = h_j t^s$ and the coefficients V_{ir} are found by the relation

$$\sum_{r=0}^{\infty} V_{ir} z^r = \exp \left(\sum_{m=1}^{\infty} \frac{h_i[-m]}{m} z^m \right). \quad (4.4)$$

The subalgebra $\mathcal{W}({}^L\mathfrak{g})$ is invariant under the action of the derivation $D = -d/dt$ defined in (3.4). This follows from the easily verified relations

$$V_i D = (D + h_i[-1]) V_i, \quad i = 1, 2.$$

To write down explicit generators of $\mathcal{W}({}^L\mathfrak{g})$ introduce elements $g_1[r]$ and $g_2[r]$ of $U(t^{-1}\mathfrak{h}[t^{-1}])$ by

$$g_1[r] = h_1[r] + \frac{1}{3} h_2[r] \quad \text{and} \quad g_2[r] = \frac{1}{3} h_2[r], \quad r < 0.$$

Moreover, we will regard the derivation $-d/dt$ as a differential operator and denote it by τ to distinguish it from the derivation D . In other words, we consider the algebra which is isomorphic to the tensor product $U(t^{-1}\mathfrak{h}[t^{-1}]) \otimes \mathbb{C}[\tau]$ as a vector space, with the relations

$$\tau H[r] - H[r] \tau = -r H[r-1], \quad H \in \mathfrak{h}.$$

A version of the Miura transformation of type G_2 was produced in [21]; cf. [3]. In the above notation write the product

$$\begin{aligned} & (\tau - 2g_1[-1] - g_2[-1])(\tau - g_1[-1] - 2g_2[-1])(\tau - g_1[-1] + g_2[-1]) \\ & \times \tau (\tau + g_1[-1] - g_2[-1])(\tau + g_1[-1] + 2g_2[-1])(\tau + 2g_1[-1] + g_2[-1]) \end{aligned} \quad (4.5)$$

as a polynomial in τ , so that it equals

$$\tau^7 + w_2 \tau^5 + \cdots + w_6 \tau + w_7, \quad w_i \in U(t^{-1}\mathfrak{h}[t^{-1}]). \quad (4.6)$$

By the results of [21], all coefficients w_i belong to $\mathcal{W}({}^L\mathfrak{g})$, and the elements $D^k w_2$ and $D^k w_6$ with $k \geq 0$ are algebraically independent generators of $\mathcal{W}({}^L\mathfrak{g})$.

The first claim can be verified directly by re-writing the screening operators in terms of the elements $g_i[r]$. We have the formulas (up to overall scalar factors):

$$V_1 = \sum_{r=0}^{\infty} V_{1r} \left(\frac{\partial}{\partial g_1[-r-1]} - \frac{\partial}{\partial g_2[-r-1]} \right),$$

where

$$\sum_{r=0}^{\infty} V_{1r} z^r = \exp \left(\sum_{m=1}^{\infty} \frac{g_1[-m] - g_2[-m]}{m} z^m \right),$$

and

$$V_2 = \sum_{r=0}^{\infty} V_{2r} \left(\frac{\partial}{\partial g_1[-r-1]} - \frac{2\partial}{\partial g_2[-r-1]} \right),$$

where

$$\sum_{r=0}^{\infty} V_{2r} z^r = \exp \left(\sum_{m=1}^{\infty} \frac{3 g_2[-m]}{m} z^m \right).$$

We show that the polynomial (4.5) is annihilated by V_1 and V_2 by using the relations

$$V_1 \tau = (\tau + g_1[-1] - g_2[-1]) V_1 \quad \text{and} \quad V_2 \tau = (\tau + 3 g_2[-1]) V_2.$$

In particular, $V_1 w_2 = V_2 w_2 = 0$, where the coefficient w_2 in the expansion (4.6) is found by

$$w_2 = -6 (g_1[-1]^2 + g_1[-1]g_2[-1] + g_2[-1]^2 - 3g_1[-2] - 2g_2[-2]). \quad (4.7)$$

We will use the standard notation $w'_2 = Dw_2$, $w''_2 = D^2w_2$, etc. for its derivatives. We have the relations

$$\begin{aligned} 2 w_3 &= 5 w'_2, \\ 4 w_4 &= w_2^2 + 12 w''_2, \\ 4 w_5 &= 3 w_2 w'_2 + 8 w'''_2. \end{aligned} \quad (4.8)$$

Our second main result provides the Harish-Chandra images of the Segal–Sugawara vectors for \mathfrak{g} produced in Sec. 3.2. We use the isomorphism of Proposition 2.4 to identify the Chevalley presentation of \mathfrak{g} with its matrix presentation. The triangular decomposition of \mathfrak{g} is defined by the subalgebras \mathfrak{n}_+ and \mathfrak{n}_- given in (2.17). For the elements of the Cartan subalgebra we have

$$G_{11}[r] \mapsto 3g_1[r] \quad \text{and} \quad G_{22}[r] \mapsto 3g_2[r], \quad r < 0.$$

Theorem B. *For the images under the isomorphism (4.2) we have*

$$\begin{aligned} S_2 &\mapsto -6 w_2, \\ S_3 &\mapsto 18 w'_2, \\ S_4 &\mapsto 9 w_2^2 - 36 w''_2, \\ S_5 &\mapsto -63 w_2 w'_2 + 72 w'''_2, \\ S_6 &\mapsto 162 w_6 - \frac{33}{2} w_2^3 + \frac{63}{2} (w'_2)^2 + 90 w_2 w''_2 - 576 w_2^{(4)}. \end{aligned}$$

Proof. The first image is easily verified using the commutations relations of $\widehat{\mathfrak{g}}$. Since (4.2) is a differential algebra isomorphism, the second image follows from the formula $S_3 = -3S'_2$, where we extend the derivative notation $S' = DS$ to elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$. The remaining relations are verified by a computer program within the FORM; see [26]. Namely, the Segal–Sugawara vectors S_a are written as linear combinations of monomials in the

generators $G_{ij}[r]$. The program first provides the respective linear combinations of ordered monomials, where the total ordering on the generators is chosen in a way consistent with the triangular decomposition of \mathfrak{g} . The resulting Harish-Chandra images $\mathfrak{f}(S_a)$ are linear combinations of the monomials containing only the diagonal generators $G_{11}[r]$ and $G_{22}[r]$. The program provides an explicit linear combination for $\mathfrak{f}(S_a)$ in terms of monomials in the generators of $\mathcal{W}({}^L\mathfrak{g})$. \square

Remark 4.1. For verification purposes, we used another program to make sure that all images $\mathfrak{f}(S_a)$ with $a = 2, \dots, 6$ do belong to the classical \mathcal{W} -algebra $\mathcal{W}({}^L\mathfrak{g})$. This was done by applying the screening operators to check that $V_1 \mathfrak{f}(S_a) = V_2 \mathfrak{f}(S_a) = 0$. \square

Corollary 4.2. *We have the relations*

$$\begin{aligned} S_3 &= -3 S_2', \\ 4 S_4 &= S_2^2 + 24 S_2'', \\ 4 S_5 &= -7 S_2 S_2' - 48 S_2'''. \end{aligned}$$

Proof. This is immediate from Theorem B and relations (4.8). \square

5 Commutative subalgebras and Gaudin model

Here we consider applications of Theorems A and B to explicit constructions of maximal commutative subalgebras of $U(\mathfrak{g})$ and to the Gaudin model associated with the simple Lie algebra \mathfrak{g} of type G_2 .

5.1 Quantization of the shift of argument subalgebras

We will use the matrix presentation of the Lie algebra \mathfrak{g} so that it is spanned by the entries G_{ij} of the matrix G given in (2.11). For any element $\mu \in \mathfrak{g}^*$ we set $\mu_{ij} = \mu(G_{ij}) \in \mathbb{C}$ so that we can regard $\mu = [\mu_{ij}]$ as a matrix of the form

$$\mu = \sum_{i,j=1}^7 e_{ji} \otimes \mu_{ij}.$$

The bilinear form (2.10) allows us to identify \mathfrak{g}^* with \mathfrak{g} . An element $\mu \in \mathfrak{g}^* \cong \mathfrak{g}$ is called *regular*, if the centralizer \mathfrak{g}^μ of μ in \mathfrak{g} has minimal possible dimension. This minimal dimension coincides with the rank of \mathfrak{g} and so equals 2.

The next theorem is implied by Theorem A and a positive solution of Vinberg's quantization problem [27] given by Rybnikov [25] and Feigin, Frenkel and Toledano Laredo [10] with the use of the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$; cf. [13] and [17] for such applications in classical types.

For a given arbitrary element μ introduce polynomials in a variable z with coefficients in $U(\mathfrak{g})$ by the formulas

$$\begin{aligned} A(z) &= \text{tr}(G + \mu z)^2, \\ B(z) &= \text{tr}(G + \mu z)^6 + 5 \text{tr} G(G + \mu z)^4 + 14 \text{tr} G(G + \mu z)^3 + 31 \text{tr} G^2(G + \mu z)^2, \end{aligned}$$

and write

$$\begin{aligned} A(z) &= A_0 z^2 + A_1 z + A_2, \\ B(z) &= B_0 z^6 + B_1 z^5 + B_2 z^4 + B_3 z^3 + B_4 z^2 + B_5 z + B_6. \end{aligned}$$

Theorem C. *For any $\mu \in \mathfrak{g}^*$, all elements A_i and B_j of $U(\mathfrak{g})$ pairwise commute. Moreover, if μ is regular, then the elements $A_1, A_2, B_1, B_2, B_3, B_4, B_5, B_6$ are algebraically independent and generate a maximal commutative subalgebra of $U(\mathfrak{g})$.*

Proof. Given a nonzero $z \in \mathbb{C}$, the mapping

$$\varrho_{\mu,z} : U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(\mathfrak{g}), \quad G_{ij}[r] \mapsto G_{ij}z^r + \delta_{r,-1}\mu_{ij}, \quad (5.1)$$

defines an algebra homomorphism. The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$ so that the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under $\varrho_{\mu,z}$ is a commutative subalgebra of $U(\mathfrak{g})$. This image is independent of z and will be denoted by \mathcal{A}_μ . For the images of the generator matrices we have

$$z \varrho_{\mu,z}(G[-1]) = G + \mu z \quad \text{and} \quad z^r \varrho_{\mu,z}(G[-r]) = G \quad \text{for } r \geq 2.$$

Consider the elements $S_2, S_6 \in \mathfrak{z}(\widehat{\mathfrak{g}})$ provided by Theorem A. All coefficients of the polynomials in z defined by

$$z^2 \varrho_{\mu,z}(S_2) \quad \text{and} \quad z^6 \varrho_{\mu,z}(S_6)$$

belong to the commutative subalgebra \mathcal{A}_μ . Moreover, $A(z) = z^2 \varrho_{\mu,z}(S_2)$, while $z^6 \varrho_{\mu,z}(S_6)$ equals $B(z)$ plus a linear combination of the terms $\text{tr} G^3$, $\text{tr} G^2$ and $\text{tr} G^2(G + \mu z)$. However, using (3.9) and the easily verified relation $(G^2)^t = G^2 + 12G$, we get

$$\text{tr} G^3 = -6 \text{tr} G^2 \quad \text{and} \quad \text{tr} G^2(G + \mu z) = -6 \text{tr} G(G + \mu z).$$

Therefore, the constant term of the linear combination is proportional to A_2 , whereas the coefficient of z is proportional to A_1 . This proves the first part of the theorem.

The second part will follow by considering the symbols \overline{A}_i and \overline{B}_j of the coefficients A_i and B_j in the associated graded algebra $\text{gr } U(\mathfrak{g}) \cong S(\mathfrak{g})$. They are given by

$$\begin{aligned} \text{tr}(\overline{G} + \mu z)^2 &= \overline{A}_0 z^2 + \overline{A}_1 z + \overline{A}_2, \\ \text{tr}(\overline{G} + \mu z)^6 &= \overline{B}_0 z^6 + \overline{B}_1 z^5 + \overline{B}_2 z^4 + \overline{B}_3 z^3 + \overline{B}_4 z^2 + \overline{B}_5 z + \overline{B}_6. \end{aligned}$$

It was shown in [10] that if the element μ is regular, then the coefficients $\overline{A}_1, \overline{A}_2, \overline{B}_1, \dots, \overline{B}_6$ are algebraically independent generators of the Poisson commutative subalgebra $\overline{\mathcal{A}}_\mu$ of $S(\mathfrak{g})$, known as the *Mishchenko–Fomenko subalgebra* or *shift of argument subalgebra*. This also follows from the earlier results of Bolsinov [2]. Furthermore, it was shown in [23] that $\overline{\mathcal{A}}_\mu$ is maximal Poisson commutative. This implies the second statement of the theorem. \square

Remark 5.1. As an additional test, we employed a computer program within the FORM [26] (similar to the one used in the proof of Theorem A) to verify some particular cases of Theorem C. Namely, we considered diagonal matrices $\mu = \text{diag}[\mu_1, \mu_2, \mu_3, 0, -\mu_3, -\mu_2, -\mu_1]$ with $\mu_1 + \mu_2 + \mu_3 = 0$ and regarded μ_1 and μ_2 as variables. The elements A_2 and B_6 are central, while A_1 and B_1 belong to the Cartan subalgebra of \mathfrak{g} and so commute with all other elements B_i . Our program verified that B_2 commutes with each of B_3, B_4 and B_5 , and that B_3 commutes with B_4 and B_5 . \square

5.2 Eigenvalues of the Gaudin Hamiltonians

A connection of the center at the critical level $\mathfrak{z}(\widehat{\mathfrak{g}})$ with the Gaudin Hamiltonians was first observed by Feigin, Frenkel and Reshetikhin [9]. They used the Wakimoto modules over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ to calculate the eigenvalues of the Hamiltonians on the Bethe vectors of the Gaudin model associated with an arbitrary simple Lie algebra \mathfrak{g} . Given an element $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$, the eigenvalues are expressed in terms of the Harish-Chandra image of S ; see also more recent work [10] for generalizations to non-homogeneous Hamiltonians.

We will use the explicit formulas for elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ provided by Theorem A and the general result of [10, Theorem 6.7] to write explicit Gaudin operators and their eigenvalues on Bethe vectors for the simple Lie algebra \mathfrak{g} of type G_2 .

Using coassociativity of the standard coproduct on $U(t^{-1}\mathfrak{g}[t^{-1}])$ defined by

$$\Delta : G_{ij}[r] \mapsto G_{ij}[r] \otimes 1 + 1 \otimes G_{ij}[r], \quad r < 0,$$

for any $\ell \geq 1$ we get the homomorphism

$$U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(t^{-1}\mathfrak{g}[t^{-1}])^{\otimes \ell} \quad (5.2)$$

as an iterated coproduct map. Now fix distinct complex numbers z_1, \dots, z_ℓ and let u be a complex parameter. Applying homomorphisms of the form (5.1) to the tensor factors in (5.2), we get another homomorphism

$$\Psi : U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(\mathfrak{g})^{\otimes \ell}, \quad (5.3)$$

given by

$$\Psi : G_{ij}[r] \mapsto \sum_{a=1}^{\ell} (G_{ij})_a(z_a - u)^r + \delta_{r,-1} \mu_{ij} \in U(\mathfrak{g})^{\otimes \ell},$$

where $(G_{ij})_a = 1^{\otimes(a-1)} \otimes G_{ij} \otimes 1^{\otimes(\ell-a)}$; see [25]. We will twist this homomorphism by the involutive anti-automorphism

$$\varsigma : U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(t^{-1}\mathfrak{g}[t^{-1}]), \quad G_{ij}[r] \mapsto -G_{ij}[r], \quad (5.4)$$

to get the anti-homomorphism

$$\Phi : U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(\mathfrak{g})^{\otimes \ell}, \quad (5.5)$$

defined as the composition $\Phi = \Psi \circ \varsigma$. Since the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$, the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under Φ is a commutative subalgebra $\mathcal{A}(\mathfrak{g})_\mu$ of $U(\mathfrak{g})^{\otimes \ell}$. It depends on the chosen parameters z_1, \dots, z_ℓ , but does not depend on u [25]; see also [10, Sec. 2].

For any $\lambda \in \mathfrak{h}^*$, the Verma module M_λ is defined as the quotient of $U(\mathfrak{g})$ by the left ideal generated by \mathfrak{n}_+ and the elements $h_i - \lambda(h_i)$ with $i = 1, 2$. We denote the image of 1 in M_λ by 1_λ . Given weights $\lambda_1, \dots, \lambda_\ell \in \mathfrak{h}^*$ consider the tensor product of the Verma modules $M_{\lambda_1} \otimes \dots \otimes M_{\lambda_\ell}$. Formulas for common eigenvectors (the *Bethe vectors*)

$$\phi(w_1^{i_1}, \dots, w_m^{i_m}) \in M_{\lambda_1} \otimes \dots \otimes M_{\lambda_\ell}$$

for the commutative subalgebra $\mathcal{A}(\mathfrak{g})_\mu$ in this tensor product can be found in [9] (also reproduced in [19]). They are parameterized by a set of distinct complex numbers w_1, \dots, w_m with $w_i \neq z_j$ and a collection (multiset) of labels $i_1, \dots, i_m \in \{1, 2\}$.

Suppose that $\mu \in \mathfrak{h}^*$. We regard μ as a functional on \mathfrak{g} which vanishes on \mathfrak{n}_+ and \mathfrak{n}_- . The system of the *Bethe ansatz equations* takes the form

$$\sum_{i=1}^{\ell} \frac{\lambda_i(h_{i_j})}{w_j - z_i} - \sum_{s \neq j} \frac{\alpha_s(h_{i_j})}{w_j - w_s} = \mu(h_{i_j}), \quad j = 1, \dots, m, \quad (5.6)$$

where $\alpha_1 = \alpha$ and $\alpha_2 = \beta$ are the simple roots; cf. [3].

Introduce the homomorphism from $U(t^{-1}\mathfrak{h}[t^{-1}])$ to rational functions in u by the rule:

$$\varrho : G_{ii}[-r-1] \mapsto \frac{\partial_u^r}{r!} \mathcal{G}_i(u), \quad r \geq 0, \quad i = 1, 2, \quad (5.7)$$

where

$$\mathcal{G}_i(u) = \sum_{a=1}^{\ell} \frac{\lambda_a(G_{ii})}{u - z_a} - \sum_{j=1}^m \frac{\alpha_{i_j}(G_{ii})}{u - w_j} - \mu_{ii}.$$

Consider the Segal–Sugawara vectors S_a with $a = 2, \dots, 6$ provided by Theorem A and Proposition 3.4. Their Harish–Chandra images $\mathfrak{f}(S_a) \in U(t^{-1}\mathfrak{h}[t^{-1}])$ are given in Theorem B. The composition $\varrho \circ \mathfrak{f}$ takes each Segal–Sugawara vector S_a to the rational function $\varrho(\mathfrak{f}(S_a))$ in u . Furthermore, we regard the image $\Phi(S)$ of S under the anti-homomorphism (5.5) as an operator in the tensor product of Verma modules $M_{\lambda_1} \otimes \dots \otimes M_{\lambda_\ell}$. The following is a consequence of [10, Theorems 6.5 and 6.7]; cf. [19].

Theorem D. *Suppose that the Bethe ansatz equations (5.6) are satisfied. If the Bethe vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is nonzero, then it is an eigenvector for the operator $\Phi(S_a)$ with the eigenvalue $\varrho(\mathfrak{f}(S_a))$ for each $a = 2, \dots, 6$. \square*

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